Malaysian Journal of Mathematical Sciences 12(1): 35–47 (2018)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

A Semi-Numerical Approach to Unsteady Squeezing Flow of Casson Fluid between Two Parallel Plates

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Received: 20 March 2017

Accepted: 26 November 2017

ABSTRACT

The unsteady squeezing flow of Casson fluid between parallel plates is re-investigated. The problem admits similarity transformations, there by reducing the equations to a non-linear differential equation of order four involving the parameter S is the non-dimensional squeezing number. The proposed homotopy perturbation method along with pade approximants is convenient in obtaining the solution. The series so generated yields expressions for skin friction and velocity.

Keywords: Squeezing flow, Homotopy, Pade approximants, Perturbation method.

1. Introduction

The squeezing flow between parallel plates has many industrial and biological situations. Most applications are in polymer processing, modeling of synthetics transformation etc. Many researchers have contributed their efforts towards better understanding of such phenomena. The basic work in this regard started with (Archibald (1956)) and followed by(Grimm (1976), Siddiqui and Irum (2008)). Earlier studies of this kind of problems are based on Reynolds equation inadequacy of this study has lead to squeeze film involving high velocity has been demonstrated by Jackson, Ishizawa and others (Ishizawa (1966), Jackson (1963)). The general study of the problems with full Navier-Stokes equations involved extensive numerical study requiring more computer time and memory. However, many features of this problem can be grasped by prescribing relative velocity to the plates. Similarity transformation were used to solve this type of problems(Birkhoff (1960), Yang (1958)).

Homotopy perturbation method first proposed by Ji-Huan He in 1998 (He. (2006), Liao. (2011),He. (2008)). HPM is the combination of traditional perturbation method and homotopy in topology. In 2013 Sumit gupta and his associates (Gupta et al. (2013)) applied this method for solving nonlinear wave-like equations. This method is useful for solving the different class of problems in the applied mathematics (like, functional integral equations (Abbasbandy (2007)), coupled system of reaction diffusion equation (Ganji and Sadighi (2006)), Helmholtz equation and fifth-order Kdv equation (Rafei and Ganji D (2006)), the epidemic model (Rafei et al. (2007)) etc.). Homotopy perturbation method for MHD squeezing flow between parallel plates has been successfully used by (Siddiqui and Irum (2008)).

In reality fluid models involved are non-Newtonian. The complicated rheological properties better explained by non-Newtonian fluids. One such model is known to be Casson fluid model (McDonald (1974)). Due to the nonlinearity of the equations, exact solutions are very rare. However, with assumptions, most of the time approximate solutions are obtained by numerical methods. So to deal with these problems semi-analytical methods are developed and more commonly used (Bujurke et al. (1995)).

In the present study homotopy perturbation method is effectively applied to solve the equations governing. The main advantage is that it yields a very rapid convergence of the series solution, only with the few iterations. For simple domains the HPM has advantages over pure numerical results. A single computer program gives the solution for a large range of expansion quantity.

1.1 Basic idea of homotopy perturbation method

To describe the HPM (Babolian et al. (2009), Liao. (2011))solution for the non-linear differential equation, we consider

$$D[f(\eta)] - f_1(\eta) = 0 \tag{1}$$

where D denotes the operator, $f(\eta)$ is unknown functions, η denote the independent variable and f_1 is known functions. D can be written as

$$D = L + N$$

where L is a simple linear part, N is remaining part of the equation (1).

The proper selection of L, N form the governing equations one can get the homotopy equation as follows

$$H(\Phi(\eta, q; q)) = (1-q)\big[L(\Phi, q) - L(v_0(\eta))\big] + q\big[L(\Phi, q) - f_1(\eta)\big] = 0 \tag{2}$$

where q is the embedding parameter which various from 0 to 1 and $v_0(\eta)$ is the initial guess to the (1).

We assume the solution of (2) as follows

$$\Phi(\eta, q) = \sum_{n=0}^{\infty} q^n f_n(\eta) \tag{3}$$

The solution to the considered problems is (3) at $q = 1$.

2. Formulation

Consider an incompressible flow of a Casson fluid between two parallel plate at a distance $z = \pm l(1-\alpha t)^{\frac{1}{2}} = \pm h(t)$ apart, where l is the initial position(at time $t = 0$). Further $\alpha > 0$ corresponds to squeezing motion of both plates until they touch each other at $t = \frac{1}{\alpha}$, for $\alpha < 0$ the plates leave each other and dilate.

The equation of Casson fluid is defined as

$$\tau_{ij} = \big[\mu_B + \big(\frac{p_y}{\sqrt{2\pi}}\big)^{\frac{1}{n}}\big]^n 2e_{ij} \tag{4}$$

where p_y is the yield stress, μ_B is the Casson viscosity and $\pi = e_{ij}e_{ij}$, where e_{ij} is the (i, j) components of the deformation rate.

The governing equations are

$$\frac{\delta \hat{u}}{\delta \hat{x}} + \frac{\delta \hat{v}}{\delta \hat{y}} = 0 \tag{5}$$

where $\hat{u}$ and $\hat{v}$ are velocity components along x and y axis respectively.

The general equation of motion in cartesian coordinates are

$$\rho \frac{D\hat{u}}{Dt} = \rho X_{\hat{x}} + \frac{\partial}{\partial \hat{x}}\sigma_{\hat{x}\hat{x}} + \frac{\partial}{\partial \hat{y}}\sigma_{\hat{x}\hat{y}} \tag{6}$$

$$\rho \frac{D\hat{v}}{Dt} = \rho X_{\hat{y}} + \frac{\partial}{\partial \hat{x}}\sigma_{\hat{x}\hat{y}} + \frac{\partial}{\partial \hat{y}}\sigma_{\hat{y}\hat{y}} \tag{7}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \hat{u}\frac{\partial}{\partial \hat{x}} + \hat{v}\frac{\partial}{\partial \hat{y}}$$

and

$$\sigma_{ij} = -\hat{p}\delta_{ij} + \tau_{ij}$$

$\delta_{ij} = 1$ when $i = j$ otherwise zero.

Since the body forces are neglected , $X_{\hat{x}}$ and $X_{\hat{y}}$ are zero. Substituting (4) into (6) and (7), we get

$$\rho \frac{D\hat{u}}{Dt} = \frac{\partial}{\partial \hat{x}}\big[-p + 2(\mu_B + \frac{p_y}{\sqrt{2\pi}})e_{ij}\big] + \frac{\partial}{\partial \hat{y}}\big[2(\mu_B + \frac{p_y}{\sqrt{2\pi}})e_{ij}\big]$$

$$\rho \frac{D\hat{v}}{Dt} = \frac{\partial}{\partial \hat{x}}\big[2(\mu_B + \frac{p_y}{\sqrt{2\pi}})e_{ij}\big] + \frac{\partial}{\partial \hat{y}}\big[-p + 2(\mu_B + \frac{p_y}{\sqrt{2\pi}})e_{ij}\big].$$

Simplifying the above equations one can get

$$\rho \frac{D\hat{u}}{Dt} = -\frac{\partial p}{\partial \hat{x}} + (\mu_B + \frac{p_y}{\sqrt{2\pi}})(2\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{x}\hat{y}})$$

$$\rho \frac{D\hat{v}}{Dt} = -\frac{\partial p}{\partial \hat{y}} + (\mu_B + \frac{p_y}{\sqrt{2\pi}})(\frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + 2\frac{\partial^2 \hat{v}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{x}\hat{y}}).$$

Substituting $\frac{D}{Dt}$ and simplifying we get

$$\frac{\delta \hat{u}}{\delta t} + \hat{u}\frac{\delta \hat{u}}{\delta \hat{x}} + \hat{v}\frac{\delta \hat{u}}{\delta \hat{y}} = -\frac{1}{\rho}\frac{\delta \hat{p}}{\delta \hat{x}} + \nu(1 + \frac{1}{\gamma})(2\frac{\delta^2 \hat{u}}{\delta \hat{x^2}} + \frac{\delta^2 \hat{u}}{\delta \hat{y^2}} + \frac{\delta^2 \hat{v}}{\delta \hat{x}\delta \hat{y}}) \tag{8}$$

$$\frac{\delta \hat{v}}{\delta t} + \hat{u}\frac{\delta \hat{v}}{\delta \hat{x}} + \hat{v}\frac{\delta \hat{v}}{\delta \hat{y}} = -\frac{1}{\rho}\frac{\delta \hat{p}}{\delta \hat{y}} + \nu(1 + \frac{1}{\gamma})(\frac{\delta^2 \hat{v}}{\delta \hat{x^2}} + 2\frac{\delta^2 \hat{v}}{\delta \hat{y^2}} + \frac{\delta^2 \hat{u}}{\delta \hat{x}\delta \hat{y}}) \tag{9}$$

where $\gamma = \mu_B\sqrt{2\pi}/p_y$ and ν is the kinematic viscosity of the fluid.

The boundary conditions are

$$\hat{u} = 0, \quad \hat{v} = v_w = \frac{dh}{dt} \qquad \text{at} \quad \hat{y} = h(t),$$

$$\frac{\delta \hat{u}}{\delta \hat{y}} = 0, \quad \hat{v} = 0, \quad \text{at} \quad \hat{y} = 0 \tag{10}$$

using (5) the above system of equation (8) and (9) can be simplified by using vorticity ω as follows

$$\frac{\delta\omega}{\delta t} + \hat{u}\frac{\delta\omega}{\delta\hat{x}} + \hat{v}\frac{\delta\omega}{\delta\hat{y}} = \nu(1+\frac{1}{\gamma})(\frac{\delta^2\omega}{\delta\hat{x^2}} + \frac{\delta^2\omega}{\delta\hat{y^2}}) \tag{11}$$

where

$$\omega = (\frac{\delta\hat{v}}{\delta\hat{x}} - \frac{\delta\hat{u}}{\delta\hat{y}}) \tag{12}$$

the velocity components $\hat{u}$ and $\hat{v}$ to be taken in the non-dimensional form is

$$\hat{u} = \frac{\alpha\hat{x}}{[2(1-\alpha t)]}F'(\eta) \tag{13}$$

$$\hat{v} = \frac{-\alpha l}{[2(1-\alpha t)^{\frac{1}{2}}]} \tag{14}$$

where

$$\eta = \frac{\hat{y}}{\sqrt{[l(1-\alpha t)]}} \tag{15}$$

using (13), (14) and (15) equation (11) can be simplified as Khan et al. (2014)

$$\big(1+\frac{1}{\gamma}\big)F^{iv}(\eta) - S(\eta F(\eta) + 3F'''(\eta) + F'(\eta)F''(\eta) - F(\eta)F'''(\eta)) = 0 \tag{16}$$

Boundary conditions (10) reduce to

$$F(0) = 0, \quad F''(0) = 0, \quad F(1) = 1, \quad F'(1) = 0. \tag{17}$$

where $S = \frac{\alpha l^2}{2v}$ is a non-dimensional squeeze number. The skin friction coefficient is defined as

$$(1+\frac{1}{\gamma})F''(1)$$

The differential equation (16) with (17)is usually solved by Runge-Kutta method this involves multiple integration process, because of two point nature of the boundary conditions. Besides this, the convergence of HPM is much better than the numerical schemes . Thus, the use of HPM provides an effective alternate approach.

3. Method of solution

We adopt two methods to solve the considered problem.

Method-I: Homotopy Perturbation Solution:

To construct the HPM solution to (16), first we select $L(\Phi) = (1+\frac{1}{\gamma})F^{iv}$ and $N(\Phi)$ is remaining part of the equation (16).

The homotopy equation for considered problem by choosing $v_0(\eta) = 0$ is the initial guess for (16) is selected from initial conditions and $f_1(\eta) = 0$ as follows

$$(1+\frac{1}{\gamma})\Phi^{iv} + q\big[-S(\eta\Phi + 3\Phi'' + \Phi'\Phi'' - \Phi\Phi''')\big] = 0 \tag{18}$$

Now assume the solution of (18) in the form of

$$\Phi(\eta, q) = \sum_{n=0}^{\infty} q^n u_n(\eta) \tag{19}$$

substitute (19) into (18) and equate the various powers of q to zero we will get the following set of equation

$$(1+\frac{1}{\gamma})u_0^{iv} = 0$$

$$(1+\frac{1}{\gamma})u_1^{iv} - S(\eta u_0 + 3u_0'' + u_0'u_0'' - u_0u_0''') = 0$$

$$(1+\frac{1}{\gamma})u_2^{iv} - S(\eta u_1 + 3u_1'' + u_1'u_1'' - u_1u_1''') = 0 \tag{20}$$

and so on. By using the boundary conditions

$$u_0(0) = 0, u_0''(0) = 0, u_0(1) = 1, u_0'(1) = 0$$

and

$$u_n(0) = 0, u_n''(0) = 0, u_n(1) = 1, u_n'(1) = 0 \quad \text{for} \quad n \geq 1$$

solving the equations, zeroth, first and second order solutions are as follows

$$u_0(\eta) = \frac{1}{2}(3\eta - \eta^3)$$

$$u_1(\eta) = -\frac{S\gamma(419\eta - 873\eta^3 + 504\eta^5 - 28\eta^6 - 24\eta^7 + 2\eta^8)}{6720(1+\gamma)}$$

$$u_2(\eta) = \frac{S^2\gamma^2}{9686476800(1+\gamma)^2}\Big(- 154163807\eta + 324472661\eta^3 - 188756568\eta^5 + 1677676\eta^6 + 17976816\eta^7 + 332046\eta^8 - 1441440\eta^9 - 109928\eta^{11} + 12376\eta^{12} + 168\eta^{13}\Big)$$

The approximate HPM solution for (16) is $F(\eta) = \sum_{n=0}^{\infty} u_n(\eta)$

Method-II: Finite Difference Solutions:

The equations mentioned above (16) with (17) were solved numerically by FDM to confirm the results obtained by us. Using standard finite difference method, i.e stepping from η_{j-1} to η_j, a Crank-Nicolson's scheme was used. These tridiagnal systems are easily solved to update the values on each grid point. Calculations were performed by dividing the interval into 10^4 sub intervals to find the associated parameters. These system of equations were solved using Mathematica.

4. Results and Discussion

A new type of series solution is presented for a unsteady squeezing of a Casson fluid between two parallel plates. In this method we generate higher order terms up to 15 using Mathematica. The series is obtained for skin friction and velocity profiles. Figures (fig (1) to fig(8)) are to elaborate the behavior of squeeze number S and Casson fluid parameter γ on axial velocities.

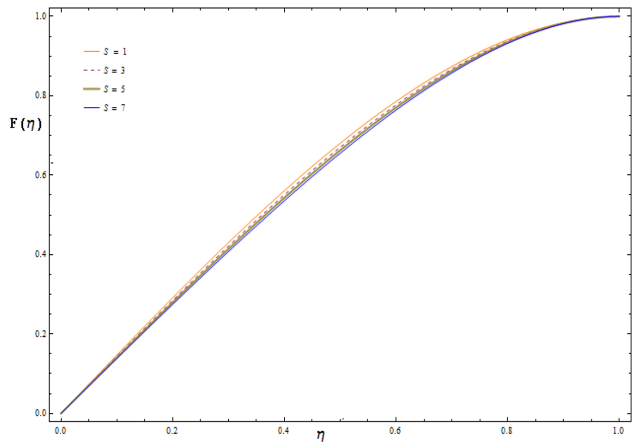

Figure 1: Effects of positive values of S on $F(\eta)$

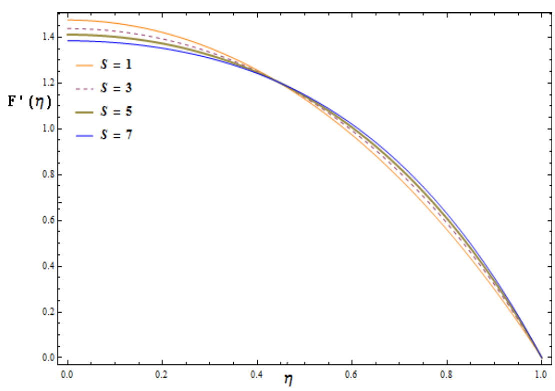

Figure 2: Effects of positive values of S on $F'(\eta)$

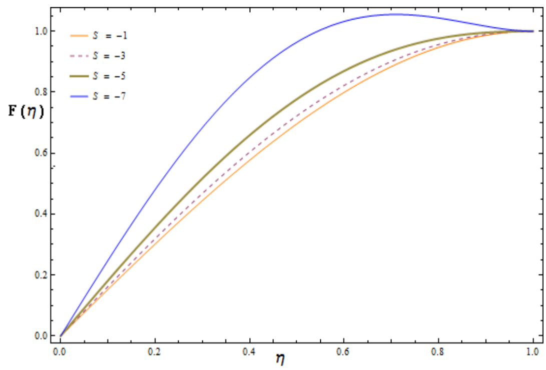

Figure 3: Effects of negative values of S on $F(\eta)$ for $\gamma = 0.8$

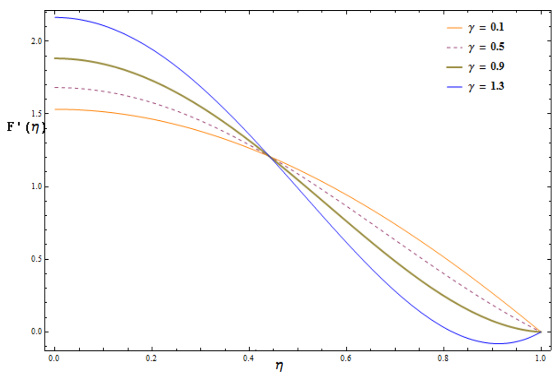

Figure 4: Effects of γ on $F'(\eta)$ for $S = -5$

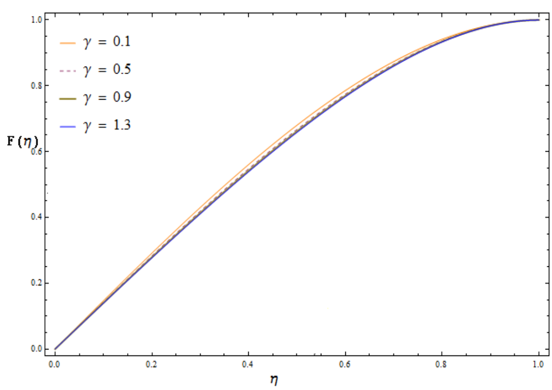

Figure 5: Effects of γ on $F(\eta)$ for $S = 5$

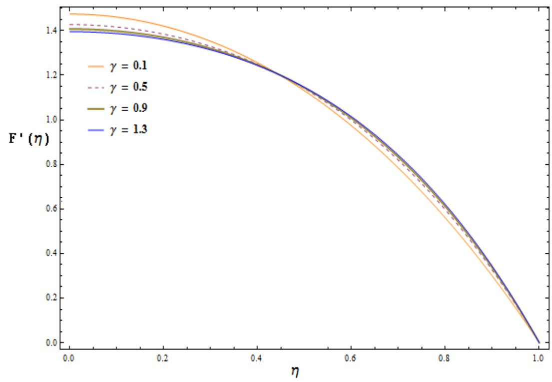

Figure 6: Effects of γ on $F'(\eta)$ for $S = 5$

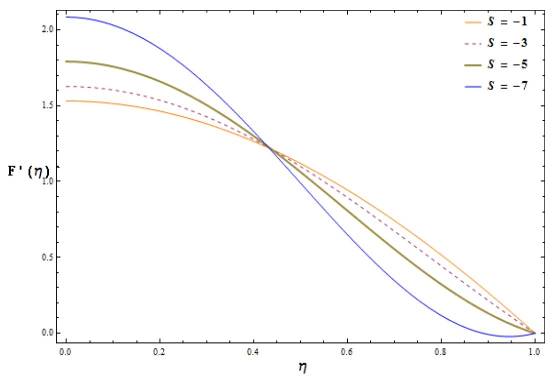

Figure 7: Effects of negative values of S on $F'(\eta)$ for $\gamma = 0.8$

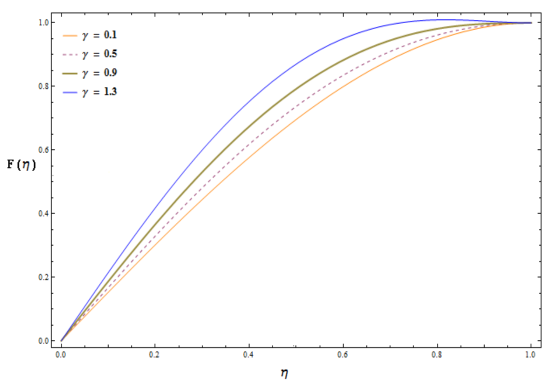

Figure 8: Effects of γ on $F(\eta)$ for $S = -5$

It is clear in fig(1) that as S increases there is a decrease in value of $F(\eta)$. Similar effect is seen in fig(2). Fig(5) and fig(6) depicts behavior of Casson fluid parameter γ on $F(\eta)$, figures (3, 4, 7 and 8) are presented to analyse the effects of parameters when plates are moving toward each other. The series is analysed and results for skin friction are shown in Table 1. The results based on the HPM agree well with the numerical results. It is observed that magnitude of the skin friction coefficient increases with increase in both non dimensional squeeze number and Casson fluid parameter. The series analysis proposed here enables us to predict the flow behavior in the range $-10 \leq S \leq 10$. Which is much better than earlier findings. A separate numerical scheme (FDM) is developed to verify the results obtained by HPM. Results are shown in Table 1.

Table 1: Comparison of results for skin friction.

S	γ	$(1+\frac{1}{\gamma})F''(1)$ Series	$(1+\frac{1}{\gamma})F''(1)$ FDM	$(1+\frac{1}{\gamma})F''(1)$(Khan et al. (2014))
-10	0.3	-1.60411	-1.58180	...
-7	0.3	-7.50738	-7.50228	...
-5	0.3	-9.73735	-9.72628	-8.73184
-3	0.3	-11.32110	-11.30390	-10.635597
-1	0.3	-12.50950	-12.49890	-12.263611
1	0.3	-13.43710	-13.42170	-13.644188
3	0.3	-14. 18310	-14.1838	-14. 976768
5	0.3	-14. 79720	-14.80740	-16. 144394
7	0.3	-15. 31230	-15.3088	.....
10	0.3	-15.9495	-15.93430	...
-2	0.1	-32. 0315	-32.0315	-31. 536967
-2	0.3	-11. 9545	-11.95450	-11. 47829
-2	0.5	-7. 88948	-7.88948	-7. 430105
2	0.1	-33.88430	-33.8843	-34.396598
2	0.3	-13. 8291	-13.8291	-14.351622
2	0.5	-9.792990	-9.79299	-10.321463

5. Conclusions

The HPM is quite flexible and can be very easily implemented on a computer compare to any other numerical methods. The analytical structure of the solution helps to determine any derived quantity, but in numerical methods an additional numerical scheme is to developed to find derived quantity, this procedure makes the things complicated. The running time and storage of HPM algorithm is very less compared to pure numerical methods.

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